A BRIEF INTRODUCTION TO SOBOLEV SPACES

In this note, we limit ourselves to the one dimensional case, since the theory for higher dimensions involves several complications. Henceforth, we use m to denote the Lebesgue measure on $\mathbb{R}$ and L^p to denote the Banach space $L^p(\mathbb{R},m)$ for any $p \in [1,\infty]$.

Weak derivatives. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function, namely a measurable function whose restriction to any compact interval is integrable. A *weak derivative* of f is a measurable function g such that

$$\int f \cdot \varphi' \, \mathrm{d}m = -\int g \cdot \varphi \, \mathrm{d}m \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\mathbb{R}),$$

where $\mathscr{C}_c^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions on $\mathbb{R}$ with compact support.

Exercise 1. (1) *Show that, if f is differentiable, then its derivative f' is a weak derivative.*

(2) *Show that the sign function*

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

is a weak derivative of $f(x) = |x|$.

Lemma 2. *If g and h are weak derivatives of a locally integrable function f, then $g = h$ almost everywhere.*

Proof. Let g and h be weak derivatives of f. From the definition, we deduce that

$$\int (g-h) \cdot \varphi \, \mathrm{d}m = 0 \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\mathbb{R}).$$

Let $[a,b]$ be any bounded interval, and let φ_ε be a mollifier as defined in the previous set of notes. Recall that we showed that $\chi_{[a,b]} * \varphi_\varepsilon$ is in $\mathscr{C}_c^\infty(\mathbb{R})$ and it converges to $\chi_{[a,b]}$ almost everywhere as $\varepsilon \to 0$. By the Dominated Convergence Theorem,

$$0 = \int (g-h) \cdot (\chi_{[a,b]} * \varphi_\varepsilon) \, \mathrm{d}m = \lim_{\varepsilon \to 0} \int (g-h) \cdot (\chi_{[a,b]} * \varphi_\varepsilon) \, \mathrm{d}m = \int (g-h) \cdot \chi_{[a,b]} \, \mathrm{d}m.$$

The fact that the integral of $g-h$ over all bounded intervals is zero implies that $g = h$ almost everywhere. □

As usual, we identify functions that are equal almost everywhere, hence, by the previous lemma, we denote by $D^j f$ the unique j-th weak derivative of f, if it exists.

Sobolev spaces.

Definition 3. *Let $p \in [1,\infty]$ and let $k \geq 0$. The* Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R}) \subset L^p$ *is the space*

$$W^{k,p} = \{f \in L^p \ : \ D^j f \in L^p \text{ for all } j \leq k\},$$

equipped with the norm

$$\|f\|_{W^{k,p}} = \sum_{j=0}^{k} \|D^j f\|_p.$$

We now show that $W^{k,p}$ is a Banach space.

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Proposition 4. *For any $p \in [1,\infty]$ and $k \geq 0$, the space $W^{k,p}$ is complete.*

Proof. Let $(f_n)_n \subset W^{k,p}$ be a Cauchy sequence. By definition of the norm, for any $j \leq k$, the sequence $(D^j f_n)_n$ is a Cauchy sequence in L^p. Since the latter is a Banach space, $(D^j f_n)_n$ converges, and let us call $g_j \in L^p$ the limit. To conclude, it suffices to prove that $D^j g_0 = g_j$, which implies that the limit g_0 of the sequence $(f_n)_n$ is in $W^{k,p}$.

Fix $j \leq k$. For any $\varphi \in \mathscr{C}_c^\infty(\mathbb{R})$, we have

$$\int g_0 \cdot D^j \varphi \,\mathrm{d}m = \lim_{n\to\infty} \int f_n \cdot D^j \varphi \,\mathrm{d}m = (-1)^j \lim_{n\to\infty} \int D^j f_n \cdot \varphi \,\mathrm{d}m = (-1)^j \int g_j \cdot \varphi \,\mathrm{d}m.$$

This shows that g_j is the j-th weak derivative of g_0 and finishes the proof. □

It is customary to write H^k for $W^{k,2}$. The space H^k is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^k} = \sum_{j=0}^{k} \langle D^j f, D^j g\rangle_{L^2} = \sum_{j=0}^{k} \int D^j f \cdot D^j g \,\mathrm{d}m,$$

or with $\overline{D^j g}$ replacing $D^j g$ above, if the functions are complex-valued.

Sobolev embeddings. Let $\alpha \geq 0$. If α is an integer, $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\mathbb{R})$ denotes the space of α-times differentiable functions. Assume now that α is not an integer, and call $\ell \geq 0$ and $\beta \in (0,1)$ its integer and fractional part respectively. We define $\mathscr{C}^\alpha$ to be the space of ℓ-times differentiable functions f whose ℓ-th derivative $D^\ell f$ is Hölder continuous of exponent β, and we equip it with the norm

$$\|f\|_{\mathscr{C}^\alpha} = \|f\|_{\mathscr{C}^\ell} + \sup_{x\neq y} \frac{|D^\ell f(x) - D^\ell f(y)|}{|x-y|^\beta}.$$

The second summand in the right-hand side above is called the β-Hölder constant of $D^\ell f$.

Theorem 5 (Sobolev Embedding Theorem)**.** *For any $p \in [1,\infty)$ and $k \geq 0$, there is a continuous injection*

$$\iota\colon W^{k,p} \hookrightarrow \mathscr{C}^\alpha, \qquad \text{where} \qquad \alpha = k - \frac{1}{p}.$$

The Sobolev Embedding Theorem follows is a consequence of the following inequality.

Lemma 6 (Morrey's Inequality)**.** *Let $f \in \mathscr{C}_c^1(\mathbb{R})$. Then $\|f\|_{\mathscr{C}^\alpha} \leq 2\|f\|_{W^{1,p}}$, where $\alpha = 1 - \frac{1}{p}$.*

Proof. Let $x < y$ be fixed. Then, since $f(y) - f(x) = \int_x^y f' \,\mathrm{d}m$, Hölder's Inequality yields

$$|f(y)| \leq |f(x)| + \int |f'| \cdot \chi_{[x,y]} \,\mathrm{d}m \leq |f(x)| + \|f'\|_p \cdot \|\chi_{[x,y]}\|_{1/\alpha} = |f(x)| + \|f'\|_p \cdot (y-x)^\alpha.$$

Averaging over $[y-1, y+1]$ with respect to x and using again Hölder's Inequality, we deduce

$$\begin{aligned} |f(y)| &\leq \frac{1}{2}\int_{y-1}^{y+1} |f(x)| + \|f'\|_p \cdot (y-x)^{1/\alpha} \,\mathrm{d}x \leq \frac{1}{2}\int_{y-1}^{y+1} |f(x)| + \|f'\|_p \,\mathrm{d}x \\ &\leq 2^{\alpha-1}(\|f\|_p + \|f'\|_p) \leq \|f\|_p + \|f'\|_p, \end{aligned}$$

which implies $\|f\|_\infty \leq \|f\|_{W^{1,p}}$.

Similarly,

$$|f(y) - f(x)| \leq \int |f'| \cdot \chi_{[x,y]} \,\mathrm{d}m \leq \|f'\|_p \cdot (y-x)^\alpha,$$

which implies that the α-Hölder constant of f is at most $\|f'\|_p$. This completes the proof. □

Proof of the Sobolev Embedding Theorem. Let us consider the case $k = 1$, and let $\alpha = 1 - \frac{1}{p}$. Morrey's Inequality implies that the inclusion

$$\iota\colon \mathscr{C}_c^1(\mathbb{R}) \hookrightarrow \mathscr{C}^\alpha$$

is continuous, if we equip the domain with the norm $\|\cdot\|_{W^{1,p}}$. Since $\mathscr{C}_c^1(\mathbb{R})$ is dense in $W^{1,p}$, the map ι extends to a linear and continous injection between $W^{1,p}$ and $\mathscr{C}^\alpha$.

The general case follows similarly by considering $D^j f$ instead of f for all $j < k$. The details are left as an exercise to the reader. □